

Infinite self-gravitating systems and cosmological structure formation

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Abstract. The usual thermodynamic limit for systems of classical self-gravitating point particles becomes well defined, as a *dynamical* problem, using a simple physical prescription for the calculation of the force, equivalent to the so-called “Jeans’ swindle”. The relation of the resulting intrinsically out of equilibrium problem, of particles evolving from prescribed uniform initial conditions in an infinite space, to the one studied in current cosmological models (in an expanding universe) is explained. We then describe results of a numerical study of the dynamical evolution of such a system, starting from a simple class of infinite “shuffled lattice” initial conditions. The clustering, which develops in time starting from scales around the grid scale, is qualitatively very similar to that seen in cosmological simulations, which begin from lattices with applied correlated displacements and incorporate an expanding spatial background. From very soon after the formation of the first non-linear structures, a spatio-temporal scaling relation describes well the evolution of the two-point correlations. At larger times the dynamics of these correlations converges to what is termed “self-similar” evolution in cosmology, in which the time dependence in the scaling relation is specified entirely by that of the linearized fluid theory. We show how this statistical mechanical “toy model” can be useful in addressing various questions about these systems which are relevant in cosmology. Some of these questions are closely analagous to those currently studied in the literature on long range interactions, notably the relation of the evolution of the particle system to that in the Vlasov limit and the nature of approximately quasi-stationary states.

INTRODUCTION

The evolution of systems of a very large number of classical point particles interacting solely by Newtonian gravity is a paradigmatic problem for the statistical physics of long range interacting systems. It is also a very relevant limit for real physical problems studied in astrophysics and cosmology, ranging from the formation of galaxies to the evolution of the largest scale structures in the universe. Indeed current theories of the universe postulate that most of the particle-like clustering matter in the universe is “cold” and “dark”, i.e. very non-relativistic and interacting essentially only by gravity, which means that the purely Newtonian approximation is very good one over a very great range of temporal and spatial scales. In practice, as we discuss further below, the very large numerical simulations used to follow the evolution of the structures in the universe, starting from the initial conditions on the density perturbations inferred from measurements of the fluctuations in the cosmic microwave background, are completely Newtonian. While simulations in this context have developed impressively in size and sophistication over the last three decades, the results they provide remain essentially phenomenological in the sense that our analytical understanding of them is very limited.

Our first aim in this contribution is to clarify, in a statistical physics language, precisely what the relevant problem is which is currently studied in the context of the problem of structure formation in cosmology. The essential point is that it corresponds simply to a specific infinite volume limit of the Newtonian problem. A well defined limit of the resulting equations of motion is given by the case that the universe does not expand. This case, we explain, corresponds to the most natural definition of the usual thermodynamic limit of the Newtonian problem. We note that this limit is distinct from the “mean-field” or “dilute” one sometimes considered¹ in the literature, and in which, under certain circumstances, one can define thermodynamic equilibria. In the limit we consider the system has no thermodynamic equilibrium, and the system is always intrinsically out of equilibrium.

The main body of this contribution will then be given over to the study of this (usual) thermodynamic limit of gravity, for a particular set of initial conditions. We will describe and characterise the evolution of clustering as observed in a set of numerical simulations, summarising the results of a series of recent publications with our collaborators [1, 2, 3]. Further we will highlight the fact that this evolution is qualitatively very similar to that seen in cosmological simulations (in an expanding universe). It is characterised notably by the fact that it is “hierarchical”, which means that the non-linear structures form at a continually increasing scale, propagating towards larger scales at a rate which is fixed by the spectrum of initial fluctuations in the initial conditions. The evolution is observed also to tend asymptotically to what cosmologists call a “self-similar” behaviour, which means that the correlations manifest a simple spatio-temporal scaling behaviour. While the temporal part of this evolution may be explained by a simple linear fluid model for the growth of perturbations, the form of the spatial part remains, as in the cosmological model, unpredicted.

We will then outline a model we have developed which describes very well the evolution of our simulations at early times. It is a two phase model, linking a perturbative treatment of the evolution at early times to an approximation in which the force on particles is dominated essentially by their nearest neighbour.

In the last part we will turn to the relevance of our analysis to open unresolved questions in the cosmological problem. Indeed one of our motivations has been that we may be able to use our system, which is relatively simple compared to a cosmological simulation, in this way as a kind of “toy model”. We focus here specifically on the “problem of discreteness” in cosmological simulations: these simulations simulate a number of particles which is many orders of magnitude less than that in the physical problem, and the question is how this limits the resolution of the corresponding results. Stated more formally, the desired physical limit is the appropriate Vlasov-Poisson system, and the problem is to understand the finite N effects². We discuss what we learn about this problem from the case of the shuffled lattice.

In our conclusions we suggest some directions for future work. In particular we discuss briefly the interest of relating our results to now very popular phenomenological

¹ See contribution of P.H. Chavanis.

² We will explain below that in the infinite volume limit it is, more precisely, a question of the relation between the evolution of the particle system, with finite particle *density*, and that in the Vlasov limit.

models for the evolution of cosmological simulations, so-called “halo models”. The central objects in this description (halos) have, apparently, very similar properties to those observed to be formed through the violent relaxation of finite self-gravitating systems. We note that the nature of these states of the finite system is analogous to the quasi-stationary states observed and much studied in one dimensional toy models³.

NEWTONIAN GRAVITY IN THE INFINITE VOLUME LIMIT

Let us start from a system of N identical point particles interacting by purely Newtonian gravity, i.e., with equations of motion

$$\ddot{\mathbf{r}}_i = -Gm \sum_{j \neq i} \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} \quad (1)$$

where dots denote derivatives with respect to time, m is the mass of the particles, G is Newton’s constant, and \mathbf{r}_i is the position of the i -th particle. If N is finite the evolution is well defined⁴, whether the system is open (i.e. in an infinite space), or enclosed in a region (with some appropriate boundary conditions).

Let us consider now the usual thermodynamic limit, i.e.,

$$N \rightarrow \infty, V \rightarrow \infty, \text{ at } n_0 = \frac{N}{V} = \text{constant} \quad (2)$$

For example we can consider taking N points which we randomly distribute in the volume V , increasing the number N in strict proportionality with the volume V . A little consideration of Eq. (1) shows that there is a problem when we take the limit: the sum on the right hand side, giving the gravitational acceleration of any given particle, is not unambiguously defined by this limiting procedure. To see this more explicitly, let us suppose that we choose our volume to be a spherical volume $V(R_s, \mathbf{r}_0)$ centred on some arbitrary fixed point, at vector position \mathbf{r}_0 , and of radius R_s , and that we take the infinite volume limit by increasing this radius $R_s \rightarrow \infty$. The force per unit mass for the point \mathbf{r}_i is then

$$\mathbf{F}(\mathbf{r}_i) = -Gm \lim_{R_s \rightarrow \infty} \sum_{j \neq i, j \in V(R_s, \mathbf{r}_0)} \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}. \quad (3)$$

³ See the contribution of S. Ruffo in this volume.

⁴ This is true of course assuming that $\mathbf{r}_i \neq \mathbf{r}_j$ for all $i \neq j$. Starting from initial conditions in which this is true — as we always shall here — this undefinedness of the problem associated with this singularity of the force is never relevant, in a finite time. Whether this singularity is “left naked” or removed by a smoothing at some scale is, however, of course relevant to the very long time behaviour of the system. Indeed the so-called “gravothermal catastrophe” in these systems arises from the divergence of the microcanonical entropy due to configurations in which pairs of particles approach one another arbitrarily close, allowing other particles in the system to explore an unlimited phase space (in momentum). We will not explore the question here of the behaviour of these systems in the limit that $t \rightarrow \infty$. See further comments below when we discuss the use of a regularisation at small scales of the force in numerical simulations.

Defining $n_i(\mathbf{r}) = \sum_{j \neq i} \delta(\mathbf{r} - \mathbf{r}_j)$, and $\delta n_i(\mathbf{r}) = \sum_{j \neq i} \delta(\mathbf{r} - \mathbf{r}_j) - n_0$ [where $\delta(\mathbf{r})$ is the Dirac delta function], this can be rewritten as

$$\begin{aligned} \mathbf{F}(\mathbf{r}_i) &= -Gm \lim_{R_s \rightarrow \infty} \int_{V(R_s, \mathbf{r}_0)} d^3x n_i(\mathbf{r}) \frac{\mathbf{r}_i - \mathbf{r}}{|\mathbf{r}_i - \mathbf{r}|^3} \\ &= -Gm \lim_{R_s \rightarrow \infty} \left[n_0 \int_{V(R_s, \mathbf{r}_0)} \frac{\mathbf{r}_i - \mathbf{r}}{|\mathbf{r}_i - \mathbf{r}|^3} + \int_{V(R_s, \mathbf{r}_0)} d^3r \delta n_i(\mathbf{r}) \frac{\mathbf{r}_i - \mathbf{r}}{|\mathbf{r}_i - \mathbf{r}|^3} \right] \end{aligned} \quad (4)$$

Force due to mean density

The first term inside the brackets, representing the force per unit mass on the particle due to the mean density n_0 on a particle inside the sphere, gives, using Gauss' theorem, for any point $\mathbf{r}_i \in V(R_s, \mathbf{r}_0)$

$$\mathbf{F}_0(\mathbf{r}_i) = -\frac{4\pi G n_0 m}{3} (\mathbf{r}_i - \mathbf{r}_0). \quad (5)$$

This result is independent of R_s , and thus is also the expression of this contribution to the force in the limit that $R_s \rightarrow \infty$. We see, however, that it depends explicitly on the choice of origin. We will analyse the second term in the square brackets in Eq. (4), due to the correction to uniformity, in the next subsection. We will verify that this latter contribution may converge to a well defined value, independently of how the limit is taken, and thus that it cannot remove this dependence of the full force on the chosen point.

To make the problem of Newtonian gravity in the usual thermodynamic limit well defined we need a uniquely defined force acting on particles. Given the arbitrariness in the result of the above calculation, we must thus give a prescription for calculating this force. As the ambiguity arises from the uniform component of the particle density, an evident choice for such a prescription is one which makes the contribution of the mean density zero for any particle. This is attained by any prescription for the sum in which the force on any given particle is determined by summing in a *symmetric* manner about that particle, e.g.,

$$\mathbf{F}(\mathbf{r}_i) = -Gm \lim_{R_s \rightarrow \infty} \sum_{j \neq i, j \in V(R_s, \mathbf{r}_i)} \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} \quad (6)$$

$$= -Gm \lim_{R_s \rightarrow \infty} \left[n_0 \int_{V(R_s, \mathbf{r}_i)} \frac{\mathbf{r}_i - \mathbf{r}}{|\mathbf{r}_i - \mathbf{r}|^3} + \int_{V(R_s, \mathbf{r}_i)} d^3r \delta n_i(\mathbf{r}) \frac{\mathbf{r}_i - \mathbf{r}}{|\mathbf{r}_i - \mathbf{r}|^3} \right] \quad (7)$$

$$= -Gm \lim_{R_s \rightarrow \infty} \left[\int_{V(R_s, \mathbf{r}_i)} d^3r \delta n_i(\mathbf{r}) \frac{\mathbf{r}_i - \mathbf{r}}{|\mathbf{r}_i - \mathbf{r}|^3} \right]. \quad (8)$$

where $V(R_s, \mathbf{r}_i)$ is the spherical volume of radius R_s *centred on the i -th point*. The force on a particle is then that due only to the fluctuations around the mean density.

This prescription for the calculation of the force is essentially equivalent to what is known as the ‘‘Jeans’ swindle’’ [4, 5]. To treat the dynamics of perturbations to an infinite self-gravitating pressureful fluid, Jeans ‘‘swindled’’ by perturbing about the state

of constant mass density and zero velocity, which is not in fact a solution of the fluid equations he analysed. Formally the “swindle” involves removing the mean density from the Poisson equation, so that the gravitational potential is sourced only by fluctuations to this mean density. This is evidently equivalent to the prescription we have just given for the gravitational force, as seen explicitly in Eq. (8).

A very nice discussion of the mathematical basis of this “Jeans’ swindle” is given by Kiessling[6], who emphasises that the presentation of what Jeans did as a “mathematical swindle” is misleading. The “swindle” can in fact be given a perfectly firm mathematical basis when it is understood, as we have presented it above, as a physical regularisation of the *force*. The mathematical inconsistency in Jeans’ analysis arises because it is done in terms of potentials which are badly defined, even in the thermodynamic limit as we have defined it. In Ref. [6] Kiessling shows that the Jeans swindle corresponds to a regularisation of the force given by the prescription used above in Eq. (6), which just corresponds to the physical prescription that the force on all particles be zero for an infinite constant density. Kiessling notes that an equivalent more physical form of the prescription is

$$\mathbf{F}(\mathbf{r}_i) = -Gm \lim_{\mu \rightarrow 0} \sum_{j \neq i} \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} e^{-\mu |\mathbf{r}_i - \mathbf{r}_j|} \quad (9)$$

where the sum now extends over the infinite space⁵, i.e., the value of the gravitational force in the thermodynamic limit is given as the limit of a screened gravitational force. One may thus actually consider the dynamics we describe below as characterising that of a system of particles with a screened gravitational force, but where the screening scale ($= \mu^{-1}$) is much larger than the length scales on which “non-linear” clustering develops (see below).

Force due to fluctuations

Having given the prescription which makes the force zero in the usual thermodynamic limit for the case of an exactly uniform distribution of matter, we must consider also whether this prescription makes the force well defined in a distribution of particles, in which there are necessarily deviations from perfect uniformity, due to the discrete nature of the particles. To answer this question we must evidently consider *how* these particles are distributed in the infinite space, i.e., the nature of the fluctuations which they give rise to. For any distribution which one proposes to study as initial conditions of the infinite self-gravitating, one must ensure that the force is then well defined.

To prescribe (and describe) how particles are distributed in infinite space we use the language of stochastic point processes, i.e., we consider the points to be distributed by a stochastic process, with certain properties. We assume only that these processes have

⁵ This means that the thermodynamic limit has been taken before the limit in μ .

only the following properties (which we require to employ them usefully here):

- *spatial ergodicity*, i.e., the value of relevant quantities characterizing their fluctuations/correlations in the infinite volume limit may be recovered by taking the ensemble average.
- *statistical homogeneity*, i.e., averages are invariant under any rigid translation of the system.
- *uniformity*, i.e., they have a well defined *non-zero* mean density, $n_0 = \lim_{N,V \rightarrow \infty} (N/V) = \langle n(\mathbf{x}) \rangle$ (where $\langle \dots \rangle$ denotes the ensemble average).

In this framework the force on a particle, and also the force field (per unit mass) at any point in space, are stochastic variables. Let us consider the latter quantity⁶, denoting it $\mathbf{f}(\mathbf{r})$. It may be written as an integral over the stochastic variable $n(\mathbf{r})$ as

$$\mathbf{f}(\mathbf{r}) = -Gm \lim_{R_s \rightarrow \infty} \left[\int_{V(R_s, \mathbf{r})} d^3 r' n(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right]. \quad (10)$$

The question of the well-definedness of the force may then be phrased as follows: is the probability distribution function (PDF) of the variable \mathbf{f} , defined at any finite R_s , well defined in the limit $R_s \rightarrow \infty$? An answer may be given as follows. Using the convolution theorem for Fourier transforms (FT) we can infer that⁷

$$|\tilde{\mathbf{f}}^2(\mathbf{k})| \propto k^{-2} |\tilde{\delta} n(\mathbf{k})|^2 \quad (11)$$

where the tilde denote the FT (and we have used the fact that the FT of $\mathbf{x}/|\mathbf{x}|^3$ is proportional to $\mathbf{k}/|\mathbf{k}|^2$). The square of the variance of any stochastic variable is, up to an appropriate normalisation in the volume⁸, its power spectrum (or structure factor). The integral of this quantity, it can be shown easily (see e.g. [7]) is proportional to the one point variance of the variable, from which it follows here that

$$\langle \mathbf{f}^2 \rangle \propto \int d^3 k k^{-2} P(\mathbf{k}) \quad (12)$$

where $P(k)$ is the power spectrum for the density field $n(\mathbf{r})$ of the point process. If this variance of the force is finite, the force PDF is necessarily well defined. This means that a *sufficient* condition for the force to be defined in the usual thermodynamic limit is just that $k^{-2} P(\mathbf{k})$ be integrable. Since we are interested here only in the possible divergences in all these quantities due to the long distance behaviour of the fluctuations⁹,

⁶ We analyse here, for simplicity, the *unconditioned* force field, i.e., the force per unit mass at an arbitrary spatial point, rather than the force on an actual point of the distribution. These quantities will in general be different, notably for the small scale properties related to divergences of the force at zero separation. For the study of convergence/divergence of the force due to the long-range nature of the force, the quantities are equivalent.

⁷ To do this FT, which is valid only in the sense of distributions, it is more appropriate to write the thermodynamic limit as in Eq. (9).

⁸ The exact definition will be given in the next section.

⁹ Note that we have already implicitly made this assumption in writing Eq. 10, as we have ignored the divergences when \mathbf{x} falls on an occupied point if there is no regularisation of the force at zero separation.

it is therefore sufficient that we require $\lim_{k \rightarrow 0} kP(\mathbf{k}) = 0$. Thus it follows that the force is well defined in the infinite volume limit *if $P(\mathbf{k})$ diverges more slowly at small k than k^{-1}* . This can be compared with the condition of uniformity of the point process, which requires that $P(\mathbf{k})$ be integrable, and thus only that it not diverge faster than k^{-3} as $k \rightarrow 0$. We can conclude that the force PDF is, at least, well defined for this sub-class of all uniform point processes.

A Poissonian point process (i.e. particles thrown randomly in the infinite volume with any finite mean density n_0), which corresponds to a constant $P(\mathbf{k})$, should thus give a well defined force in the thermodynamic limit. The PDF of the force on a particle for this case (with the same implicit regularisation we have described) was first calculated by Chandrasekhar in the forties[8], and gives the so-called Holtzmark distribution (see [7] for the full expression and a simplified derivation). The PDF is indeed well defined, and thus the dynamical problem in the thermodynamic limit with the given prescription. The variance is, in fact, infinite due to the small scale behaviour of the force: the probability at very large forces decays slowly (as a power-law) due to the unbounded growth of the interparticle force in configurations in which two particles are arbitrarily close. Imposing a cut-off regulating the divergence in the force at small separations, it is simple to determine (see [7]) how the variance of the force, which is then finite, diverges as this cut-off goes to zero. We will consider in detail below the case of “shuffled lattice” initial conditions. A detailed treatment of the statistics of the force field in these distributions is given in Ref. [9]. The power spectrum $P(k)$ in this case (see below) is proportional to k^2 at small k , and it may be shown explicitly that the variance of the force is therefore finite in the defined thermodynamic limit, modulo divergences due to the small scale behaviour of the force. Indeed it is shown in Ref. [9] that the exact gravitational force PDF decays exponentially at large values of the field in distributions of this type in which there is zero probability of particle overlap, but as in the Poisson case (with infinite variance) if there is a non-zero probability for such configurations¹⁰.

Relation to expanding universe case

The principal conclusion of the preceding two sections is that there is a natural thermodynamic limit for the dynamical problem of self-gravitating particles in the Newtonian limit. The equations of motion of the particles are simply Eq. (1), where the sum is now prescribed to be taken symmetrically about each particle¹¹. Provided the distribution of particles considered has fluctuations which decay sufficiently rapidly at large distances, this prescription, we have seen, gives a force on particles which is well defined¹². Further, as we will see further below, the behaviour of fluctuations at

¹⁰ An analogous treatment of generic power law interactions can be found in Ref. [10].

¹¹ In practice this sum will be performed in the numerical simulations described below using the Ewald sum method. In this technique (see e.g. Ref. [11]) the long distance part of the sum is converted to a sum in reciprocal space, from which the badly defined term at $\mathbf{k} = 0$, due to the mean density, is removed.

¹² The power spectrum $P(\mathbf{k})$ is the FT of the reduced two point density-density correlation function, and thus the constraints on the sufficiently slow divergence at small k of the former correspond to constraints

arbitrarily large scales is a property of the distribution which is not modified by the gravitational evolution, so the force will then be defined at all times if it is defined in the initial distribution¹³.

The simplest way to understand the relation of this dynamical problem to that treated in cosmology is by comparing these equations of motion to those which apply in numerical simulations of structure formation in the universe. These are written (see e.g. Ref. [12])

$$\ddot{\mathbf{x}}_i + 2 \frac{\dot{a}(t)}{a(t)} \dot{\mathbf{x}}_i = - \frac{Gm}{a^3} \sum_{j \neq i} \frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|^3} \quad (13)$$

where \mathbf{x}_i are the positions (in “comoving coordinates”) of the particles, and $a(t)$ is the expansion rate of the cosmological model. The sum on the right-hand side, defining the force per unit mass on the particles, is evaluated in precisely the way we have just discussed, i.e., symmetrically, so that the force on each point is zero when we neglect the density *fluctuations* associated with the points¹⁴. The thermodynamic limit we have described for the purely Newtonian problem simply corresponds to $\dot{a}(t) = 0$, i.e. to the formally non-expanding limit of the equations.

The equations of motion Eq. (13) for the cosmological case, where $\dot{a}(t) \neq 0$ and $a(t)$ obeys the correct equation derived from the full general relativistic equations, can in fact be derived also in a purely Newtonian framework, but using a limit different to the one we have considered. To see this let us return to Eq. (1), but with the sum now evaluated as given in Eq. (3), i.e. choosing a specific centre and summing the force in spheres centred on it. Taking now only the contribution to the force due to the mean density, i.e., neglecting the effect of deviations from uniformity, Eq. (5) gives the equation of motion for any particle

$$\ddot{\mathbf{r}}_i = - \frac{4\pi G n_0 m}{3} (\mathbf{r}_i - \mathbf{r}_0) . \quad (14)$$

To derive this formula we only used the fact that n_0 is well defined at any given instant, but did not assume that it was independent of time. It follows that we may seek, consistent with our assumptions, a solution of these equations describing a simple global dilatation of the whole (spherical) system, i.e. of the form

$$\mathbf{r}_i(t) - \mathbf{r}_0 = R(t) [\mathbf{r}_i(t_0) - \mathbf{r}_0] \quad (15)$$

on the rapidity of decay of the latter at large separations. Alternatively one can also relate $P(\mathbf{k})$ to the variance of mass in a volume, the latter being given by the integral of the former weighted by the FT of the window function describing the volume (see e.g. Ref. [7]). For $P(k \rightarrow 0) \sim k^n$, with $n < 1$, this gives, for example, that the variance in particle number in spheres of radius decays as R^{3-n} . We have therefore shown above that the gravitational force has a finite variance due to fluctuations at large distances if the variance of these fluctuations grows more slowly with R than R^4 .

¹³ The linearized fluid theory (see below) describes the evolution fluctuations at large scales, giving a simple k -independent evolution for the power spectrum at small k . The latter thus maintains its initial functional form.

¹⁴ In practice, just as in the simulations we will describe below, the sum is normally implemented numerically using an Ewald summation method.

where $R(t_0) = 1$. In this case $R(t)$ must then satisfy the equation

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\rho(t) = -\frac{4\pi G}{3}\frac{\rho_0}{R^3}, \quad (16)$$

where we have defined $\rho(t) = mn_0(t)$, the mean mass density at time t , and $\rho_0 = \rho(t_0) = mn_0(t_0)$. This equation may be integrated directly to give

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\left[\frac{\rho_0}{R^3} + \frac{\kappa}{R^2}\right] \quad (17)$$

where κ is a constant of the motion, proportional to the conserved “energy”:

$$\frac{8\pi G}{3}\kappa = \frac{1}{2}\dot{R}^2 - \frac{GM_0}{R}. \quad (18)$$

where M_0 is a mass density proportional to the (constant) mass inside the sphere defined by any particle and the origin of the coordinates. The dynamics of the dilatation is clearly equivalent to that of a single particle in a central gravitational field, with $\kappa = 0$ determining the critical escape velocity at the initial time.

These equations are identical to the Friedmann equations, derived within the framework of general relativity (see e.g. [13]) and which describe an expanding (or contracting) universe, for the case that the expansion (or contraction) is sourced only by non-relativistic matter. In practice all current standard-type cosmological models have in common that the universe is described very well by these equations (i.e. in which the observed expansion of the universe is driven only by non-relativistic matter¹⁵) in the epoch at which structure formation develops. The constant $R(t)$ and κ acquire a different physical meaning to that ascribed to them in the Newtonian derivation — the former describes stretching in time of the spatial sections of the four dimensional space-time and the latter the curvature of these spatial sections — but this difference in interpretation is of no relevance in the treatment of the problem of structure formation.

To recover the full equations used in this context, i.e. Eq. (13), it suffices now to change coordinates, defining the comoving coordinates of any particle by

$$\mathbf{r}_i(t) - \mathbf{r}_0 = R(t)\mathbf{x}_i(t) \quad (19)$$

Substituted in Eq. (1) this gives

$$\ddot{\mathbf{x}}_i + \frac{\dot{R}}{R}\dot{\mathbf{x}}_i = -\lim_{R_s \rightarrow \infty} \frac{Gm}{R^3} \sum_{j \neq i, j \in V(R_s, 0)} \frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|^3} - \frac{\ddot{R}}{R}\mathbf{x}_i \quad (20)$$

$$= -\frac{Gm}{R^3} \left[\lim_{R_s \rightarrow \infty} \sum_{j \neq i, j \in V(R_s, 0)} \frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|^3} - \frac{4\pi}{3}n_0\mathbf{x}_i \right] \quad (21)$$

$$= -\frac{Gm}{R^3} \lim_{R_s \rightarrow \infty} \sum_{j \neq i, j \in V(R_s, \mathbf{x}_i)} \frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}. \quad (22)$$

¹⁵ We will remark below on the addition of a “cosmological constant”, which is now believed to cause an acceleration of the observed expansion at recent epochs.

where here the sum over $j \in V(R_s, \mathbf{x})$ is over all the particles in the comoving sphere of radius R_s (i.e. over the physical sphere of radius $R(t)R_s$ at time t) centred at the comoving coordinate position \mathbf{x} . Making the identification of $R(t)$ with $a(t)$, this last form is precisely Eq. (13), with the prescription for the summation of the force made explicit. Note that we have used Eq. (16) in the passage between Eq. (20) and Eq. (21), and in passing from Eq. (21) to Eq. (22) we have assumed that the force due to the fluctuations (i.e. once the term due to the mean density is subtracted) is well defined independently of how the sum is taken. The criterion for this are identical to those discussed above.

The only missing element in this derivation with respect to the full cosmological simulations currently studied is the co-called “cosmological constant”. This enters only as a modification of the Friedmann equation and thus of the function $a(t)$ in the Eq. (13). We note that it in fact can be derived in a completely Newtonian framework, by generalising the above treatment with an additional classical repulsive force acting between all pairs of particles and linearly proportional to their vector separation. Because of its linearity it plays no role in the evolution of fluctuations, other than that encoded in the modification it causes of the uniform expansion.

In summary the equations considered in cosmology to model formation of structure in the universe can be understood in a purely Newtonian framework as those describing the dynamics of particles, enclosed in a very large sphere and initially almost uniformly distributed with velocities close to those described by a homogeneous dilatation of this sphere (modelling the Hubble expansion). Formally we can consider the limit of the corresponding equations of motion in which there is no expansion. Physically this limit corresponds not to a very large static sphere — there is no static solution for such a configuration — but to a different regularisation of the gravitational force, which we have explained is naturally considered as the usual thermodynamic limit of the Newtonian problem of particles distributed uniformly in an infinite space.

In this precise sense one can therefore study the problem of cosmological structure formation in the non-expanding limit. It is a natural starting point for a statistical mechanics approach to this problem, and as we will see one which already has most of the qualitative features of the cosmological problem.

Relation to other $N \rightarrow \infty$ limits

It is useful to comment on the relation of the (usual) thermodynamic limit we have discussed to other limiting procedures used for the description of self-gravitating systems studied in the literature. More specifically we have in mind two such limits:

- the “dilute” thermodynamic limit, and
- the Vlasov-Poisson limit.

The first of these limits arises in the context of the well-known treatment of gravity in the framework of the microcanonical ensemble, which has been discussed by a series of authors, going back to Emden in the early twentieth century (see e.g. [14] for a review, or the contribution of Chavanis in this volume). Such a treatment of a self-gravitating

system of particles becomes possible if the particles are enclosed in a finite box and a cut-off is introduced to regularise the gravitational potential at the origin. Equilibrium states, i.e. states maximising the entropy exist for sufficiently large energy, which may then survive as meta-stable equilibria for the case that the cut-off goes to zero. This treatment becomes formally valid in the limit

$$N \rightarrow \infty, V \rightarrow \infty, \text{ at } \frac{N}{V^{1/3}} = \text{constant}. \quad (23)$$

The mean mass and number density thus scale as

$$\rho_0 \propto n_0 = \frac{N}{V} \propto V^{-2/3}. \quad (24)$$

This scaling is clearly different to the limit we have taken, at fixed mean mass and number density. This means that the results obtained from this microcanonical treatment are not expected to be relevant to the dynamics we will study. This can be seen by noting that the characteristic time scale for the dynamical evolution of the systems we study (and which we will in fact use as our time unit below), is the *dynamical time*, defined as

$$\tau_{\text{dyn}} \equiv \frac{1}{\sqrt{4\pi G \rho_0}}. \quad (25)$$

This *physical* time scale¹⁶ thus goes to infinity in the “dilute” thermodynamic limit, i.e. the dynamics we describe never takes place. Alternatively we can say that the effects described by this “dilute” limit are strictly relevant to the systems we study only on times scales which are infinitely long compared to those which we consider. Indeed the physics described by this microcanonical treatment is understood to be relaxational, characterised by time scales which diverge with N when expressed in terms of the dynamical time-scales. As we are studying already an infinite N limit, this relaxational time scale diverges, and any result from this treatment of a finite, closed, system cannot describe the physics of our infinite system, of which any finite subsystem is always open¹⁷.

The Vlasov-Poisson limit, on the other hand, is a limit which has been studied for finite systems (see e.g. [15]). Its validity as an approximation to the full dynamics of an N -body system has been rigorously demonstrated [16], for gravity *with a regularisation* of the singularity in the potential, when the limit $N \rightarrow \infty$ is taken at fixed volume and a fixed time, with the scaling of the particle mass $m \propto 1/N$. This latter scaling makes the acceleration given by the sum on the right hand side in Eq. (1) converge to the appropriate mean field value. In this case we therefore have

$$n_0 = \frac{N}{V} \propto N \text{ and } \rho_0 = \frac{mN}{V} = \text{constant}. \quad (26)$$

¹⁶ E.g. for a mass density of one gram per cm³, one finds $\tau_{\text{dyn}} \approx 18.2$ minutes.

¹⁷ This treatment may, nevertheless, be relevant, qualitatively, to understanding aspects of the very long time evolution of the systems we study. Indeed, as we will describe, highly clustered regions can behave over much longer timescales than τ_{dyn} as roughly independent from the rest of the system, and so may manifest long-time relaxational behaviors like those of isolated finite systems which are, at least partially, described by this microcanonical treatment.

Again this is clearly a different limit to the one we have considered. We see, however, that this limit leaves invariant the characteristic time scale τ_{dyn} , which we noted is that of the dynamics for the infinite systems we will describe and analyse below. The reason for this is in fact that this time scale emerges from a treatment of this dynamics which is derived by treating it in a Vlasov-Poisson approximation. In this case, however, this approximation now applies to the infinite system, i.e., in which the limit $N \rightarrow \infty$ has already been taken, but as we have described, at constant n_0 and ρ_0 . Although there is no rigorous demonstration in the literature of the validity of such a limit for such a system, it is clear that it must correspond, if it indeed exists, to the limit

$$n_0 \rightarrow \infty \text{ at } \rho_0 = mn_0 = \text{constant}. \quad (27)$$

Indeed when the Vlasov limit is taken in a fixed (finite) volume it may be written in this volume-independent way, which thus generalizes without difficulty to the infinite volume case. As we will discuss further below, in cosmology this limit is of great importance, as the *aim* of the numerical simulations of self-gravitating Newtonian particle systems in this context is to reproduce this limit as well as possible.

Lastly we emphasize an important point concerning the usual thermodynamic limit for self-gravitating particles: we have shown that this limit exists for the *dynamical problem*. It is well known, and indeed one of the starting points of any discussion of long-range interactions, that in this limit the use of the usual instruments of equilibrium thermodynamics is problematic. Indeed the gravitational potential violates both the essential conditions necessary for their use formulated by Ruelle[17] — the condition of “temperedness” due to the slow decay of the potential (i.e. slower than the dimensionality of space) at large separations, and the condition of stability due to the behaviour at small separations. Our treatment of the problem in the usual thermodynamic limit is completely consistent with this. Indeed it turns that the dynamical problem we have formulated is an intrinsically out of equilibrium one, the evolution of the system remaining always explicitly time dependent (in a calculable manner, determined as we will see, by the analysis in the Vlasov-Poisson limit). Thus any treatment of the problem as an equilibrium one, or a quasi-equilibrium one (see e.g. the contribution of Saslaw in this volume) will at best be a useful, but non-rigorous, approximation.

EVOLUTION FROM SHUFFLED LATTICE INITIAL CONDITIONS: PHENOMENOLOGY

We describe in this section, in a phenomenological manner, the evolution under their self-gravity of particles initially placed in a shuffled lattice (SL) configuration as observed in a set of numerical simulations. This synthesizes the salient results of a more detailed study reported in [1]. Having given the exact definition of a SL, and explained our choice of this specific class of initial conditions, we will present the results of simulations, focusing on the standard two point characterisations of the observed evolving correlations, in terms of the reduced two point correlation function and its reciprocal

space equivalent, the power spectrum¹⁸. In the presentation of these results we will highlight the qualitative similarity with the known behaviour of analogous cosmological simulations, and explain for uninitiated readers the essential theoretical results — just the original analysis of Jeans of the evolution of density perturbations in the fluid limit — used to understand them.

Shuffled Lattices

We use the term SL to refer to the infinite point distribution obtained by randomly perturbing a perfect cubic lattice: each particle on this lattice, of lattice spacing ℓ , is moved randomly (“shuffled”) about its lattice site *independently* of all the others. A particle initially at the lattice site \mathbf{R} is then at $\mathbf{x}(\mathbf{R}) = \mathbf{R} + \mathbf{u}(\mathbf{R})$, where the random vectors $\mathbf{u}(\mathbf{R})$ are specified by $p(\mathbf{u})$, the PDF for the displacement of a single particle.

An ensemble of SL configurations is thus characterised by the lattice spacing ℓ and the PDF $p(\mathbf{u})$. We will use in the simulations below a simple top-hat form, i.e., $p(\mathbf{u}) = (2\Delta)^{-3}$ for $\mathbf{u} \in [-\Delta, \Delta]^3$, and $p(\mathbf{u}) = 0$ otherwise¹⁹. Taking $\Delta \rightarrow 0$, at fixed ℓ , one thus obtains a perfect lattice, while taking $\Delta \rightarrow \infty$ at fixed ℓ , gives an uncorrelated Poisson particle distribution[7]. Once the form of the PDF is specified the class of SL we consider is thus a two parameter family, characterized by ℓ and Δ . We refer to the latter as the *shuffling parameter*. Using the top-hat PDF it is simple to verify that it is simply the square root of the variance of the displacement PDF, i.e., $\Delta^2 = \int d^3u \mathbf{u}^2 p(\mathbf{u})$. It is convenient to define also the dimensionless ratio $\delta \equiv \Delta/\ell$, which we will refer to as the *normalized shuffling parameter*.

To characterize the correlation properties of the SL let us consider its power spectrum. We recall that for a point (or continuous mass) distribution in a cube of side L , with periodic boundary conditions, this quantity is defined as²⁰

$$P(\mathbf{k}) = \frac{1}{L^3} \langle |\tilde{\delta}(\mathbf{k})|^2 \rangle, \quad (28)$$

where $\tilde{\delta}(\mathbf{k})$ are the Fourier components of the density contrast $\delta(\mathbf{x}) \equiv (\rho(\mathbf{x}) - \rho_0)/\rho_0$ (and $\rho(\mathbf{x})$ is the mass density), i.e.,

$$\tilde{\delta}(\mathbf{k}) = \int_{L^3} \delta(\mathbf{x}) \exp(-i\mathbf{k} \cdot \mathbf{x}) d^3\mathbf{x}. \quad (29)$$

for $\mathbf{k} = (2\pi/L)\mathbf{n}$ where \mathbf{n} is a vector of integers. The infinite volume limit is then obtained by sending $L \rightarrow \infty$, keeping the average mass (or number) density fixed. We give the finite volume form for $P(\mathbf{k})$ here as we will in fact treat in our simulations an infinite system defined in this way.

¹⁸ We will recall the definitions of these quantities and some of their essential properties at the appropriate points, but not the details of how they are estimated in the simulations. These may be found in [1].

¹⁹ We will explain below that the exact form of the PDF is not of importance.

²⁰ We adopt the normalisation which is used canonically in cosmology, for which the asymptotic behaviour at large \mathbf{k} , characteristic of any point process, is $P(\mathbf{k} \rightarrow \infty) = \frac{1}{n_0}$.

It is straightforward (see Ref. [7]) to show that the power spectrum for a generic SL is given by

$$P(\mathbf{k}) = \frac{1 - |\tilde{p}(\mathbf{k})|^2}{n_0} + L^3 \sum_{\mathbf{n} \neq 0} |\tilde{p}(\mathbf{k})|^2 \delta_{\mathbf{K}}(\mathbf{k}, \mathbf{n} \frac{2\pi}{\ell}) \quad (30)$$

where $\tilde{p}(\mathbf{k})$ is the Fourier transform of the PDF for the displacements $p(\mathbf{u})$ (i.e. its characteristic function), and $\delta_{\mathbf{K}}$ is the three-dimensional Kronecker symbol. The second term in Eq. (30) is non-zero only at the corresponding discrete (non-zero) values of \mathbf{k} . It is in fact simply the power spectrum of the unperturbed lattice modulated by $|\tilde{p}(\mathbf{k})|^2$. Thus only the first term in Eq. (30) contributes around $\mathbf{k} = 0$. Taylor expanding this term, assuming only that $p(\mathbf{u})$ has finite variance, we obtain the leading small- k behavior

$$P(\mathbf{k}) \approx \frac{k^2 \Delta^2}{3n_0} = \frac{1}{3} k^2 \delta^2 \ell^5, \quad (31)$$

where $k = |\mathbf{k}|$. Note that this small- k behavior of the power spectrum of the SL therefore does not depend on the details of the chosen PDF for the displacements, but only on its (finite) variance. For this reason in fact the results we will discuss for the chosen PDF do not in fact depend in detail on this specific form²¹. Note that the power spectrum is a function only of k at small k , which means that the large scale fluctuations in the system are statistically isotropic. The behavior $P(k) \propto k^2$ means that, as one would expect, an SL is at large scales a distribution with fluctuations which are suppressed compared to those in a Poisson distribution. Indeed the SL belongs for this reason to the class of “superhomogeneous”[18] (or “hyperuniform”[19]) point processes with this property, characterized by the behavior $P(\mathbf{k} \rightarrow 0) = 0$.

Let us now comment on why we choose this particular class of SL initial conditions. The reason is that they are a very simple (two parameter) class of initial conditions resembling those used in cosmological simulations. In this context initial conditions are prepared by applying *correlated* displacements to a lattice. By doing so one can produce, to a good approximation, a desired power spectrum at small wavenumbers²². The spectra of currently favored models contain several parameters, roughly interpolating between power law forms $P(k) \propto k^n$, with n varying from $+1$ at the smallest k to close to -3 at the largest k included in the initial conditions of numerical simulations. In particular the SL allows us to mimic an important feature of these simulations, related to the problem of discreteness we will discuss further below: with the two parameters characterising the class of initial conditions, *we can modify independently the particle density and the amplitude of the power spectrum at small wavenumbers*. This can be seen from Eq. (31), as it is clear that to keep the latter fixed it suffices to impose, when varying ℓ , the condition

$$\delta^2 \ell^5 = \text{constant}. \quad (32)$$

²¹ In Ref. [2] simulations like those discussed here, but with a Gaussian PDF are analysed.

²² See [20] for a detailed analysis of the algorithm used.

TABLE 1. Details of the SL used as initial conditions in the simulations of [1]. N is the number of particles, L is the box size, ℓ the lattice constant and Δ (δ) the (normalized) shuffling parameter.

Name	$N^{1/3}$	L	ℓ	Δ	δ	m/m_{64}
SL64	64	1	0.015625	0.015625	1	1
SL32	32	1	0.03125	0.0553	0.177	8
SL24	24	1	0.041667	0.00359	0.0861	18.96
SL16	16	1	0.0625	0.00195	0.03125	64
SL128	128	2	0.015625	0.015625	1	1

Numerical Simulations

In numerical simulations we cannot treat of course a truly infinite SL. Instead we treat the infinite system defined by such a lattice, in a finite cubic box of size $L = N^{1/3}\ell$, plus periodic copies. Our results will then be representative of the thermodynamic limit only insofar as they do not depend on the size of the box. We will see below that the nature of the dynamics is such that this can be true only for a finite time, i.e., the evolution of this infinite, but periodic, system will represent that in the thermodynamic limit for a finite time. To follow the evolution for a longer time requires an ever larger box. Any effects which depend on the box size are *finite size* effects which are not related to the real behavior of the well defined physical limit we are studying²³.

In Table 1 are given the details of the SL initial conditions of the N body simulations reported in [1], and which we will discuss here.

The numerical evolution has been performed using the publicly available code GADGET [21, 22]. This is based on a tree algorithm for the calculation of the force, and allows one to perform simulations in an infinite periodic distribution using the Ewald summation method. The potential used is *exactly* equal to the Newtonian potential for separations greater than a softening length ε , and regularized below this scale²⁴. More details of tests on the accuracy of the simulations (energy conservation, robustness to change of integration parameters etc.) can be found in [1].

Let us explain the rationale for the choice of the parameters in Table 1. Firstly, we have chosen our (arbitrary) units of length, mass and time as follows. Our unit of length is given by the box side of the SL64 simulation and our unit of mass by the particle mass in this same simulation. The particle masses are then chosen so that the mass density ρ_0 is constant, which means that the dynamical time defined above in Eq. (25) is the same in all simulations. This is convenient because this is, as we have mentioned above, an appropriate characteristic time-scale for the evolution which can be derived from the fluid limit (see below). We have chosen a smoothing parameter $\varepsilon = 0.00175$ in all simulations. It is therefore in all cases significantly smaller — at least a factor of

²³ A good analogy is in the simulation of the out of equilibrium dynamics of glassy systems, e.g., the ordering dynamics of a quenched ferromagnet.

²⁴ We do not give here the rather complicated functional form of this smoothing, which can be found in [21, 22]. The important point here is that the smoothed force goes to zero continuously as particles approach one another.

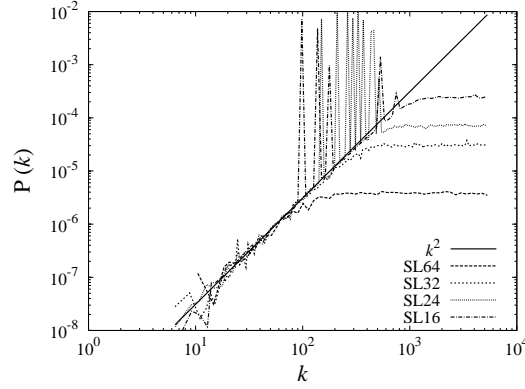


FIGURE 1. The power spectra (averaged in spherical shells) of the SL configurations specified above in Tab. 1 as a function of the modulus of \mathbf{k} . The solid line is the theoretical ($\propto k^2$) behavior for small k given by Eq. (31). At large k , the four power spectra are equal to $1/n_0$, with the corresponding value of n_0 . The peaks arise from the second term in Eq. (30). From Ref. [1].

ten — than the lattice spacing (which is close in all cases to the initial average distance between nearest neighbors). The box size is the same in all but one simulation. This latter simulation (SL128), which is the biggest one, is used to test the accuracy with which our results are representative of the thermodynamic limit. Thus it is chosen to have the same parameters to SL64, differing only in its volume (which increases by a factor of 8). Finally the values of δ have been chosen for each ℓ so that the condition in Eq. (32) is fulfilled. The measured power spectra of realizations of SL with the parameters given in Table 1 are shown in Fig. 1. We see, up to statistical fluctuations, that the spectra are indeed of the same amplitude at small k . Note that the curve corresponding to each initial condition is most easily identified by the fact that the asymptotic (large \mathbf{k}) level of the power spectrum is inversely proportional to the mean particle density.

Lastly, the particles are assigned zero velocity in the initial conditions (at $t = 0$), and they are run in each case until a time when the finite size of the box manifestly becomes important (see below).

Let us comment further on the use of the smoothing parameter ε . The goal here is to study the evolution of a distribution of self-gravitating *point* particles, and the reason for the introduction of ε is that it allows a much more rapid numerical integration. It does this because it modifies the trajectories in which particles have close encounters, which in lead to very large accelerations (and thus the necessity for very small time steps). The use of the smoothing is only justified therefore if the associated physical effects are not important for the macroscopic properties we will study. A priori we do not know whether this is the case, and we simply test numerically (see [1]) for the robustness of our results to changes, towards significantly smaller values, of the smoothing parameters. We interpret the observed ε -independence of our results as meaning that the artificial effects it introduces in the dynamics may be relevant to the quantities we measure only on time scales longer than those we can study in our simulations.

We mention in this context the following point, to which we will return further below. In cosmological simulations of particle systems the smoothing is understood as a physical parameter, introduced with the intention of making the particle system

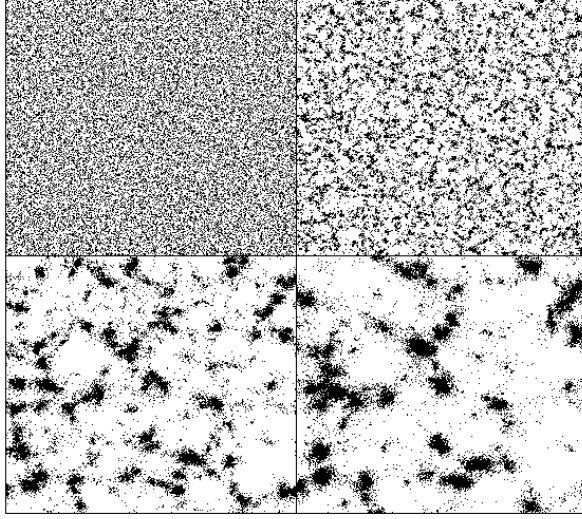


FIGURE 2. Snapshots of SL64, at $t = 0$, and the evolved configurations obtained at subsequent times, $t=3, 6, 8$. These are projections on the $z = 0$ plane.

behave more like the desired Vlasov-Poisson limit (in which two body encounters are neglected). In practice, however, values of the ratio ε/ℓ even smaller than those adapted here are typically used. Thus the parameter choices we have made here would be considered as perfectly reasonable if our goal were, as in cosmology, to reproduce the Vlasov-Poisson limit (and, as far as possible, nothing else).

Results

In Fig. 2 are shown four snapshots²⁵ of the simulation SL64. We see visually that non-linear structures (i.e. regions of strong clustering) appear to develop first at small scales, and then propagate progressively to larger scales. Eventually the size of the structures become comparable to the box-size. From this time on the evolution of the system will no longer be representative of the thermodynamic limit we are studying. Up to close to this time, however, it is indeed the case that all the properties we will study below show negligible dependence on the box size (see Refs. [1, 2] for more detail).

²⁵ The results taken from Ref. [1] are given in units which are not exactly those of dynamical time, but rather in which $\tau_{\text{dyn}} = 1.092$. This corresponds to time units of 1000 seconds if the mass density were 1g.cm^{-3} .

Evolution of correlations in real space

We consider now the evolution of clustering in real space, as characterized by the reduced correlation function $\xi(r)$. We recall that this is simply defined as

$$\xi(\mathbf{r}) = \langle \delta(\mathbf{x} + \mathbf{r}) \delta(\mathbf{x}) \rangle , \quad (33)$$

where $\langle \dots \rangle$ is an ensemble average, i.e., an average over all possible realizations of the system (and we have assumed statistical homogeneity). It is useful to note that this can be written, for $r \neq 0$, and averaging over spherical shells,

$$\xi(r) = \frac{\langle n(r) \rangle_p}{n_0} - 1 , \quad (34)$$

where $\langle n(r) \rangle_p$ is the *conditional average density*, i.e., the (ensemble) average density of points in an infinitesimal shell at distance r from a point of the distribution. Thus $\xi(r)$ measures clustering by telling us how the density at a distance from a point is affected, in an average sense, by the fact that this point is occupied. In distributions which are statistically homogeneous the power spectrum $P(\mathbf{k})$ and $\xi(\mathbf{r})$ are a Fourier conjugate pair (see e.g. Ref. [7]).

The correlation functions we will discuss below will invariably be monotonically decreasing functions of r . It is then natural to define the scale λ by

$$\xi(\lambda) = 1 . \quad (35)$$

This scale then separates the regime of weak correlations (i.e. $\xi(r) \ll 1$) from the regime of strong correlations (i.e. $\xi(r) \gg 1$). In the context of gravity these correspond, approximately, to what are referred to as the *linear* and *non-linear* regimes, as a linearized treatment of the evolution of density fluctuations (see below) is expected to be valid in the former case. Eq. (35) can also clearly be considered as a definition of the *homogeneity scale* of the system. Physically it gives then the *typical size of strongly clustered regions*.

In Fig. 3 is shown the evolution of the absolute value $|\xi(r)|$ in a log-log plot, for the SL128 simulation. These results translate quantitatively the visual impression gained above. More specifically we observe that:

- Starting from $\xi(r) \ll 1$ everywhere, non-linear correlations (i.e. $\xi(r) \gg 1$) develop first at scales smaller than the initial inter-particle distance.
- After two dynamical times the clustering develops little at scales below ε . The clustering around and below this scale is characterized by an approximate “plateau”. This corresponds to the resolution limit imposed by the chosen smoothing.
- At scales larger than ε the correlations grow continuously in time at all scales, with the scale of non-linearity [which can be defined, as discussed above, by $\xi(\lambda) = 1$] moving to larger scales.

From Fig.3 it appears that, once significant non-linear correlations are formed, the evolution of the correlation function $\xi(r)$ can be described, approximately, by a simple “translation” in time. This suggests that $\xi(r, t)$ may satisfy in this regime a spatio-temporal scaling relation:

$$\xi(r, t) \approx \Xi(r/R_s(t)) , \quad (36)$$

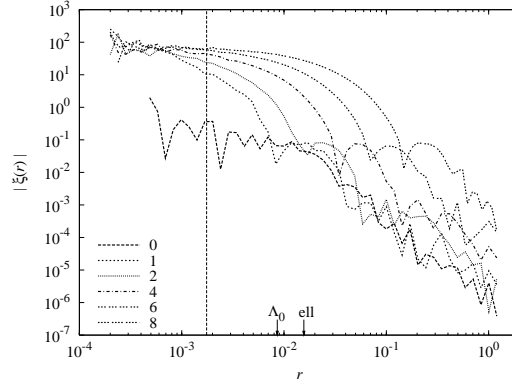


FIGURE 3. Behavior of the absolute value of the correlation function $|\xi(r)|$ in SL128 at times $t = 0, 1, 2, 4, 6, 8$. The vertical dotted line indicates ε . From [1].

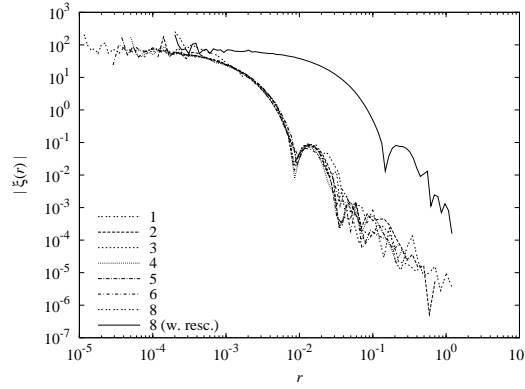


FIGURE 4. Collapse plot of $\xi(r,t)$: for each time $t > 1$ we have rescaled the x -axis by a time-dependent factor to collapse all the curves (dashed ones) to that at time $t = 1$. We have added for comparison $\xi(r,t = 8)$ without rescaling (“w. resc.”, continuous line). From [1].

where $R_s(t)$ is a time dependent length scale which we discuss in what follows. In order to see how well such an ansatz describes the evolution, we show in Fig. 4 an appropriate “collapse plot”: $\xi(r,t)$ at different times is represented with a rescaling of the x -axis by a (time-dependent) factor chosen to superimpose it as closely as possible over itself at $t = 1$, which is the time from which the “translation” appears to first become a good approximation. We can conclude clearly from Fig. 4 that the relation Eq. (36) indeed describes very well the evolution, down to separations of order ε , and up to scales at which the noise dominates the estimator (see [1] for further details). In Fig. 5 is shown the evolution of the rescaling factor $R_s(t)$ found in constructing Fig. 4, as a function of time [with the choice $R_s(1) = 1$]. Also shown is a theoretical curve which we will explain in the next subsection. Note that since we have defined the homogeneity scale λ by $\xi(\lambda) = 1$ it is clear that, once the spatio-temporal scaling relation is valid, we have $\lambda(t) \propto R_s(t)$. A fit to a simple functional form for $\Xi(r)$, a shallow power law followed by an exponential cut-off, may be found in Ref. [1].

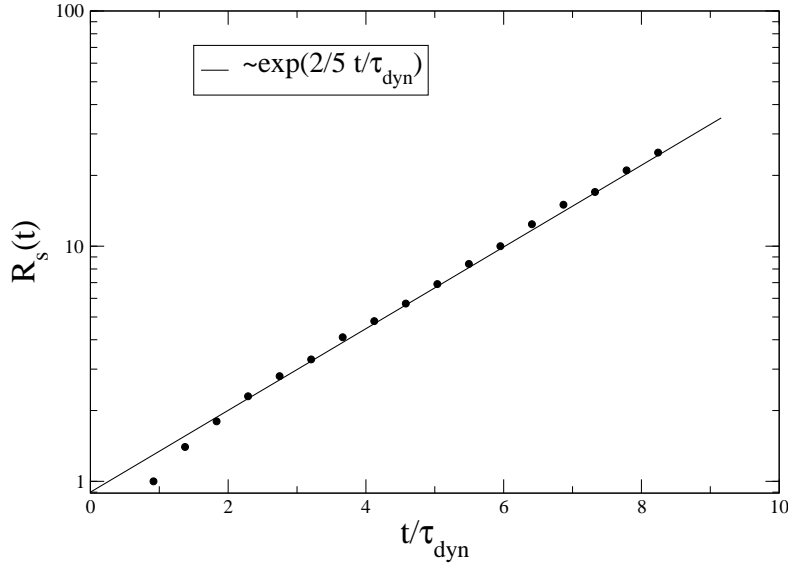


FIGURE 5. Evolution of the function $R_s(t)$ in SL128 (points) compared with its prediction from linearized fluid treatment, as explained in the next section. From [1].

Evolution of the Power Spectrum

To see how the observed temporal dependence of the correlation function is explained theoretically, in analogy to how this is done in cosmological simulations, we outline very briefly the essential result of the Jeans-type analysis of the evolution of perturbations in a self-gravitating, and pressureless, fluid. For comparison of its predictions with the numerical results for the discrete N-body system, it is simplest then to turn to the reciprocal space description of the clustering²⁶.

The equations describing the evolution of a non-relativistic pressureless self-gravitating fluid are the following²⁷

$$\begin{aligned} \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) &= 0, & \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla_{\mathbf{x}}) \mathbf{v} &= \mathbf{g} \\ \nabla_{\mathbf{x}} \cdot \mathbf{g} &= -4\pi G(\rho - \rho_0), & \nabla_{\mathbf{x}} \times \mathbf{g} &= 0, \end{aligned}$$

where $\rho(\mathbf{x}, t)$ is the mass density, $\mathbf{v}(\mathbf{x}, t)$ the velocity field and $\mathbf{g}(\mathbf{x}, t)$ the gravitational field. These equations can also be obtained by an appropriate truncation of Vlasov-Poisson equations. Linearizing in the density perturbations, and the velocity, one obtains

$$\ddot{\delta}(\mathbf{x}, t) = 4\pi G \rho_0 \delta(\mathbf{x}, t) \text{ and } \ddot{\tilde{\delta}}(\mathbf{k}, t) = 4\pi G \rho_0 \tilde{\delta}(\mathbf{k}, t). \quad (37)$$

²⁶ The real and reciprocal space description are, of course, fundamentally equivalent. The convenience of one space over another is simply in how effects due to their estimation in finite samples enter.

²⁷ As mentioned above a rigorous derivation of these equations in the thermodynamic limit as we have defined it can be found in Ref. [6].

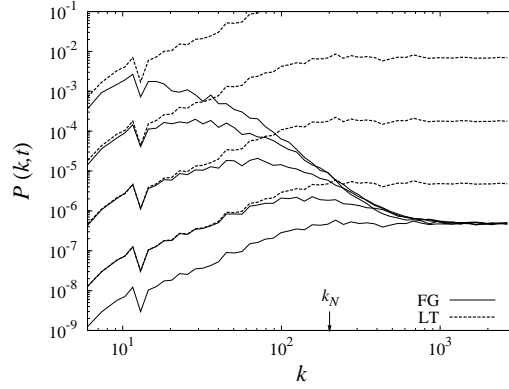


FIGURE 6. Evolution of the power spectrum in SL128 (solid lines — label FG): the curves are for time equal to 0,2,4,6,8 (from bottom to up). The dashed lines labeled with LT show the predictions of fluid linear theory, i.e., Eq. (39) with $P(\mathbf{k}, 0)$ measured in the simulation at $t = 0$ for the same time steps. The arrow labeled “ k_N ” shows the value of the corresponding Nyquist frequency $k_N = \pi/\ell$. From [1].

For the case we are analysing, of zero initial velocities, therefore

$$\delta(\mathbf{k}, t) = \delta(\mathbf{k}, 0) \cosh\left(\sqrt{4\pi G \rho_0} t\right). \quad (38)$$

and thus for the power spectrum

$$P(\mathbf{k}, t) = P(\mathbf{k}, 0) \cosh^2(t/\tau_{\text{dyn}}). \quad (39)$$

The evolution of the power spectrum as estimated in SL128 is shown in Fig. 6. Along with the numerical results is shown the prediction Eq. (39). We observe that:

- The linear theory prediction describes the evolution very accurately in a range $k < k^*(t)$, where $k^*(t)$ is a wave-number which decreases as a function of time. This is precisely the qualitative behavior one would anticipate (and also observed in cosmological simulations): linear theory is expected to hold approximately only above a scale in real space, and therefore up to some corresponding wavenumber in reciprocal scale, at which the averaged density fluctuations are sufficiently small so that the linear approximation may be made²⁸. This scale in real space, as we have seen, clearly increases with time, and thus in reciprocal space decreases with time. We note that at $t = 6$ only the very smallest k -modes in the box are still in this *linear regime*, while at $t = 8$ this is no longer true (and therefore finite size effects are expected to begin to play an important role at this time).
- At very large wave-numbers, above k_N , the power spectrum remains equal to its initial value $1/n_0$. This is simply a reflection of the necessary presence of shot noise fluctuations at small scales due to the particle nature of the distribution.

²⁸ A more precise study of the validity of the linearized approximation is given in [1].

- In the intermediate range of k the evolution is quite different, and *slower*, than that given by linear theory. This is the regime of *non-linear clustering* as it manifests itself in reciprocal space.

Spatio-temporal scaling and “self-similarity”

We now return to real space and explain a little further how the spatio-temporal scaling relation Eq. (36) for the correlation function we have observed may be linked to the linearized fluid theory, and then what it tells us about the nature of the clustering in the system.

In the context of cosmological N body simulations this kind of behavior, *when $R_s(t)$ is itself a power law (in time)*, is referred to as *self-similarity*. Such behavior is expected in an evolving self-gravitating fluid (see e.g. [13, 23, 24]) because of the scale-free nature of gravity, *if the expanding universe model and the initial conditions contain no characteristic scales*, i.e., if the input model has a simple power law $P(\mathbf{k}) \propto k^n$. On theoretical grounds there are different expectations ([24, 25, 23]) about the range of exponents n of the power spectrum which should give self-similar behavior, because of how effects coming notably from particle discreteness, the finite box size and force smoothing break the scale-free idealisation. It is widely agreed that it applies well for $-1 < n < -1$ in the literature, but there are differing conclusions about the observed degree of self-similarity outside this range.

The simple arguments used in this context to predict the temporal scaling of the correlation functions can be generalized easily to the case of a non-expanding universe (which, just like the relevant cosmological models, has no characteristic scale). To do so one simply *assumes* that such a spatio-temporal scaling relation holds exactly, *i.e., at all scales*, from, say, a time $t_s > 0$. For $t > t_s$ we have then

$$P(\mathbf{k}, t) = \int_{L^3} \exp(-i\mathbf{k} \cdot \mathbf{r}) \xi(\mathbf{r}, t) d^3\mathbf{r} \quad (40)$$

$$= R_s^3(t) \int_{L^3} \exp(-iR_s(t)\mathbf{k} \cdot \mathbf{x}) \Xi(|\mathbf{x}|) d^3\mathbf{x} \quad (41)$$

$$= R_s^3(t) P(R_s(t)\mathbf{k}, t_s) . \quad (42)$$

where we have chosen $R_s(t_s) = 1$. Assuming now that the power spectrum at small k is amplified as given by linear theory, i.e., as in Eq. (39), one infers for any power spectrum $P(k) \sim k^n$ that ²⁹

$$R_s(t) = \left(\frac{\cosh \frac{t}{\tau_{\text{dyn}}}}{\cosh \frac{t_s}{\tau_{\text{dyn}}}} \right)^{\frac{2}{3+n}} \rightarrow \exp \left[\frac{2(t - t_s)}{(3 + n)\tau_{\text{dyn}}} \right] \text{ for } t \gg t_s . \quad (43)$$

²⁹ Linear theory is expected[13] in fact to describe the evolution of the power spectrum at small k only for $n < 4$. This is because any reorganisation of smaller scales itself produces a term $\propto k^4$, so that evolution on small scales is then expected to dominate the evolution of large scale modify that at small k .

In the asymptotic behavior the relative rescaling in space for any two times becomes a function only of the *difference* in time between them so that we can write

$$\xi(r, t + \Delta t) = \xi\left(\frac{r}{R_s(\Delta t)}, t\right) ; \quad R_s(\Delta t) = e^{\frac{2\Delta t}{(3+n)\tau_{\text{dyn}}}}. \quad (44)$$

Let us now return to Fig.5. In addition to the measured values of $R_s(t)$ is plotted the best-fit (corresponding to the time $t_s \sim 2.5$) given by Eq. (44), with $n = 2$ for the SL. The agreement is clearly very good. The system is thus indeed very well described by the self-similar scaling predicted by linearized fluid theory after an initial transient time.

All these behaviours of the two point correlations are qualitatively just like those observed in cosmological simulations in an expanding universe. More general than the “self-similar” properties (which apply only to power law initial conditions), the clustering can be described as “hierarchical”, a feature typical of all currently favoured cosmological (“cold dark matter” -type) models: structures develop at a scale which increases in time, at a rate which can be determined from linear theory. This is given the following physical interpretation: clustering may be understood essentially as produced by the collapse of small initial overdensities which evolve as prescribed by linear theory, independently of pre-existing structures at smaller scales, until they “go non-linear”. Thus the process is essentially characterised by a flow of power from large scales to small scales³⁰.

One of the crucial points about this description of clustering is that it tends to support the hypothesis that cosmological simulations may be understood almost entirely within a fluid description, i.e., more broadly within the framework of Vlasov-Poisson equations. We will discuss this point further below, after the next section, which shows that we can in fact understand significant aspects of the clustering we have described without assuming this framework.

EVOLUTION FROM SHUFFLED LATTICE IC: THEORY

We present in this section various analytical and numerical approaches to understanding more fully the dynamical evolution from SL initial conditions, beyond the linearized fluid theory which we have considered in the previous section.

Phase 1: “Melting” of the lattice

The evolution of a self-gravitating perturbed lattice — whether these perturbations are correlated or not — may be described using a perturbative treatment completely analogous to that used in condensed matter physics to describe classical phonons (see

³⁰ The extrapolation of this kind of picture has been used in constructing various quite successful phenomenological models which has been used to describe the non-linear clustering observed in numerical simulations, notably the so-called “halo-model” (see e.g. Ref. [26]) to which we will return below, as well as prescriptions like that of Peacock and Dodds [27] for calculating the final power spectrum.

e.g. Ref. [28]). Substituting $\mathbf{r}_i(t) = \mathbf{R} + \mathbf{u}(\mathbf{R}, t)$ directly in Eq. (1), where \mathbf{R} is the lattice vector of the i th particle and $\mathbf{u}(\mathbf{R}, t)$ its displacement, one obtains, expanding each term in the force to linear order in the displacements,

$$\ddot{\mathbf{u}}(\mathbf{R}, t) = - \sum_{\mathbf{R}'} \mathcal{D}(\mathbf{R} - \mathbf{R}') \mathbf{u}(\mathbf{R}', t). \quad (45)$$

The matrix \mathcal{D} is the *dynamical matrix*. For gravity it may be written

$$\mathcal{D}_{\mu\nu}(\mathbf{R} \neq \mathbf{0}) = Gm \left(\frac{\delta_{\mu\nu}}{R^3} - 3 \frac{R_\mu R_\nu}{R^5} \right), \quad \mathcal{D}_{\mu\nu}(\mathbf{0}) = - \sum_{\mathbf{R} \neq \mathbf{0}} \mathcal{D}_{\mu\nu}(\mathbf{R}) \quad (46)$$

where $\delta_{\mu\nu}$ is the Kronecker delta. Note that a sum over the copies, associated with the periodic boundary conditions, is implicit in these expressions.

The Bloch theorem for lattices tells us that \mathcal{D} is diagonalized by plane waves in reciprocal space. We define the discrete Fourier transform and its inverse by

$$\tilde{\mathbf{u}}(\mathbf{k}, t) = \sum_{\mathbf{R}} e^{-i\mathbf{k} \cdot \mathbf{R}} \mathbf{u}(\mathbf{R}, t) \quad (47)$$

$$\mathbf{u}(\mathbf{R}, t) = \frac{1}{N} \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{R}} \tilde{\mathbf{u}}(\mathbf{k}, t), \quad (48)$$

where the sum in Eq. (48) is over the first Brillouin zone of the lattice, i.e., for a simple cubic lattice $\mathbf{k} = \mathbf{n}(2\pi/L)$, where \mathbf{n} is a vector of integers of which each component n_i ($i = 1, 2, 3$) takes all integer values in the range $-N^{1/3}/2 < n_i \leq N^{1/3}/2$. Using these definitions in Eq. (45) we obtain

$$\ddot{\tilde{\mathbf{u}}}(\mathbf{k}, t) = -\tilde{\mathcal{D}}(\mathbf{k}) \tilde{\mathbf{u}}(\mathbf{k}, t) \quad (49)$$

where $\tilde{\mathcal{D}}(\mathbf{k})$, the Fourier transform (FT) of $\mathcal{D}(\mathbf{R})$, is a symmetric 3×3 matrix for each \mathbf{k} .

Diagonalising $\tilde{\mathcal{D}}(\mathbf{k})$ one can determine, for each \mathbf{k} , three orthonormal eigenvectors $\mathbf{e}_n(\mathbf{k})$ and their eigenvalues $\omega_n^2(\mathbf{k})$ ($n = 1, 2, 3$), which obey the Kohn sum rule

$$\sum_n \omega_n^2(\mathbf{k}) = -4\pi G \rho_0. \quad (50)$$

Given the initial displacements and velocities at a time $t = 0$, the dynamical evolution of the particle trajectories is then given as

$$\mathbf{u}(\mathbf{R}, t) = \frac{1}{N} \sum_{\mathbf{k}} [\mathcal{P}(\mathbf{k}, t) \tilde{\mathbf{u}}(\mathbf{k}, 0) + \mathcal{Q}(\mathbf{k}, t) \dot{\tilde{\mathbf{u}}}(\mathbf{k}, 0)] e^{i\mathbf{k} \cdot \mathbf{R}} \quad (51)$$

where the matrix elements of the “evolution operator” \mathcal{P} and \mathcal{Q} are

$$\mathcal{P}_{\mu\nu}(\mathbf{k}, t) = \sum_{n=1}^3 U_n(\mathbf{k}, t) (\mathbf{e}_n(\mathbf{k}))_\mu (\mathbf{e}_n(\mathbf{k}))_\nu \quad (52)$$

$$\mathcal{Q}_{\mu\nu}(\mathbf{k}, t) = \sum_{n=1}^3 V_n(\mathbf{k}, t) (\mathbf{e}_n(\mathbf{k}))_\mu (\mathbf{e}_n(\mathbf{k}))_\nu. \quad (53)$$

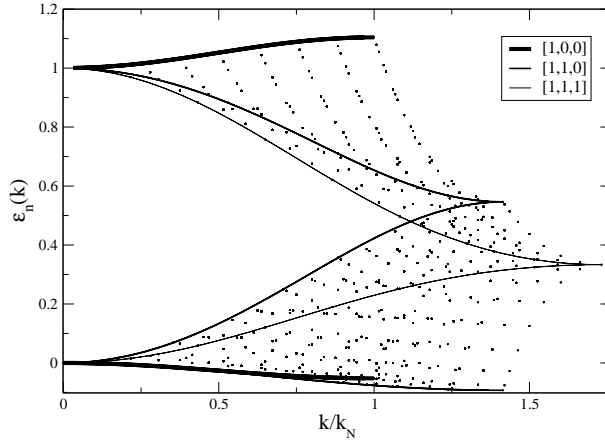


FIGURE 7. Normalized eigenvalues (see text) for the eigenvectors of the dynamical matrix of a 16^3 simple cubic lattice. From Ref. [29].

and³¹

$$U_n(\mathbf{k}, t) = \cosh\left(\sqrt{\omega_n^2(\mathbf{k})}t\right), V_n(\mathbf{k}, t) = [\sinh\left(\sqrt{\omega_n^2(\mathbf{k})}t\right)]/\sqrt{\omega_n^2(\mathbf{k})} \quad (54)$$

and the convention for the square root is: $\sqrt{\omega^2} = |\omega|$ if $\omega^2 > 0$, and $\sqrt{\omega^2} = i|\omega|$ if $\omega^2 < 0$. Thus, depending on the sign of $\omega_n^2(\mathbf{k})$ and the initial conditions, each mode is either growing/decaying or oscillatory in nature.

To fully solve for the evolution in this approximation requires thus only the determination of the eigenvectors and eigenvalues of the N 3×3 matrices $\hat{\mathcal{D}}(\mathbf{k})$. This is straightforward to do and computationally inexpensive. In Fig. 7 are shown the spectrum of the normalized eigenvalues

$$\varepsilon_n(\mathbf{k}) \equiv -\frac{\omega_n^2(\mathbf{k})}{4\pi G\rho_0}, \quad (55)$$

for a 16^3 simple lattice. Full detail on these calculations may be found in Ref. [29]. The eigenvalues are plotted as a function of the modulus k of \mathbf{k} , in units of the Nyquist frequency $k_N = \frac{\pi}{\ell}$. The fact that the eigenvalues at a given k do not have the same value is a manifestation of the anisotropy of the lattice. Shown in the plot are also lines linking eigenvectors oriented in some chosen directions. This allows one to see the branch structure of the spectrum, which is familiar in the context of analogous calculations in condensed matter physics (see e.g. [28]). Increasing particle number only changes the density of reciprocal lattice vectors in these plots (since $\mathbf{k}/k_N = \mathbf{n}/N^{1/3}$), just filling in more densely the plot of the eigenvalues in Fig. 7 but leaving its form essentially unchanged.

Formally the expansion used of the force in this treatment — which we call “particle linear theory” or PLT — is valid until the relative separation of any two particles

³¹ When this treatment is generalized (see Ref. [29]) for cosmological simulations, only these mode functions change.

becomes equal to their initial separation on the lattice, but numerical study is required to determine the usefulness in practice of the linearized approximation. In [29] we have investigated this question in detail, comparing the results of the evolution under full gravity (in numerical simulations), with that obtained using the formulae just given. It turns out that it gives an excellent description of the evolution (for the typical quantities describing clustering) until a time when the average relative displacement approaches the lattice spacing, i.e., until a significant number of particles start to approach close to one another (and the lattice “melts”).

Phase 2: Nearest neighbour domination

The PLT approximation for the force breaks down when particles approach closely nearby particles, simply because the force exerted becomes dominated by these particles. It is thus natural to consider whether this perturbative phase may be followed by one in which these interactions between nearest neighbors (NN) may be relevant in understanding the clustering which emerges. It turns out that in fact a very good approximation to the evolution of the SL is given by assuming an abrupt switch from a phase in which the forces on particles are given by the PLT approximation, to a phase in each particle feels only the force due its NN. Further this approximation works until a very significant amount of non-linear correlation has developed, essentially covering most of the transient period prior to the asymptotic self-similar scaling we have discussed above.

In Refs. [1, 2] indirect evidence was presented for the validity of such a model, using the fact the following relation was observed to hold in numerical simulations, in the relevant range of time scales:

$$\omega(r) dr = \left(1 - \int_0^r \omega(s) ds\right) \cdot 4\pi r^2 \langle n(r) \rangle_p dr, \quad (56)$$

Here $\omega(r)$ is the NN PDF, i.e., $\omega(r)dr$ is the probability that a particle has its NN in the radial shell between $[r, r + dr]$. The relation is exact[7] if all but two point correlations may be neglected, i.e., if the correlations may be fully accounted for only due to two body correlations.

In a recent paper [3] we have shown directly the validity of this two phase approximation to the evolution of an SL at early times, by comparing the full numerical evolution with gravity to a direct numerical integration of this model. Results are shown in Fig. 8. The only free parameter here is the time at which we switch from integration with PLT to that with the single NN, and the results in the figure are given for the choice of this time in each case which gives the best fit to the full evolution³². The time t_{\max} at which the comparisons are given in this figure are the latest time up to which the NN approximation gives a close fit to the full evolution, i.e., the largest time up to which this two phase model of the evolution works well in each case.

³² The values of this time determined in the way are completely consistent with the model and can be understood in greater detail by a closer analysis (see Ref. [3]).

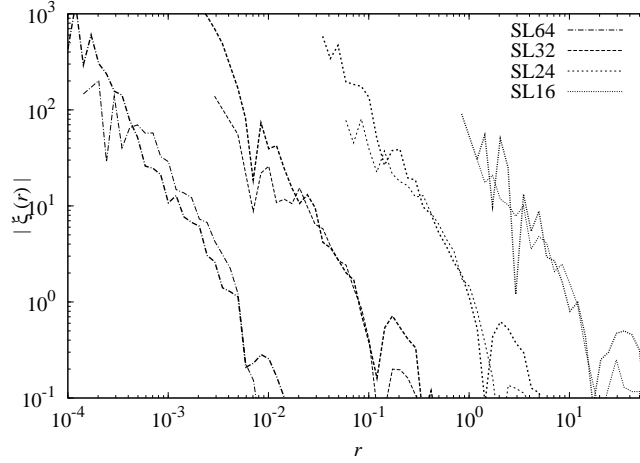


FIGURE 8. The two-point correlation function at the times $t_{\max} = 1, 2.5, 3.5, 4.5$ for the different SL initial conditions as indicated, in both the full gravity simulations (thick lines) and the simulations of the two phase model described in the text (thin lines). For clarity the x axis has been rescaled for each initial condition (as otherwise the curves are, to a very good approximation, all superimposed). From Ref. [3].

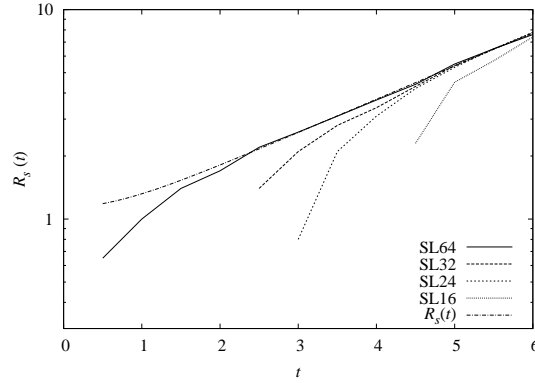


FIGURE 9. Evolution of the rescaling factor $R_s(t)$ in the different simulations. Also shown is the self-similar behavior Eq. (43). From Ref. [1].

Phase 3: Self-similar evolution

We have discussed above how the evolved SL approximates very well, at sufficiently long times, the “self-similar” scaling corresponding to the temporal behavior of the function $R_s(t)$ predicted by the linearized fluid theory. While above we showed the results for the SL128 simulation, we have carried out in Ref. [1] the same analysis for the other initial conditions, and the results are shown in Fig. 9. We see that in each case the tendency to join the asymptotic scaling behaviour is observed, but occurs at a later time. This is simply because as the initial δ decreases (i.e. the normalised shuffling, cf. Table 1) the duration of the PLT phase increases, and thus the NN phase is entered later, and breaks down later. Comparing the estimated breakdown times t_{\max} for the two phase approximation given in the caption of Fig. 8 above with Fig. 9, we see that each system is

close to joining the self-similar behavior at the time when the NN approximation breaks down.

We do not have, currently, an analytical or semi-analytical treatment which allows us to understand the essential physical question of the origin of the functional form of the *spatial dependence of the asymptotic correlation function*. However the following observation, which has been emphasized in Refs. [1, 2, 3] is important. Both Figs. 9 and 5 are derived in the approximation that the correlation function is fit by a single functional form. To the extent that such a fit is reasonably good, it means that the correlation function — in the relevant range of scale, in which $\xi(r)$ ranges from close to 10^2 down to about 0.1 — has approximately the spatial dependence of the asymptotic correlation function. This suggests strongly — but does not prove — that this asymptotic form of the non-linear correlations is in fact determined by the early time clustering, in the NN dominated phase, while its temporal behavior is determined by the fluid limit. We will discuss this point further in the next section, because it is of direct relevance to the question we consider concerning the role of discreteness in simulations of this kind.

SOME OPEN ISSUES

We have underlined the qualitative similarity of the evolution of the infinite SL configurations to that observed in numerical studies of gravitational clustering in the universe. The simulations in this context differ from those here in the initial perturbations applied to the lattice, and the fact that the expansion of the spatial background is incorporated [using the equations of motion Eq. (13) instead of Eq. (1)]. We conclude that the model we have studied, in which we neglect these extra complexities specific to cosmology, provides an interesting “toy model” to address some essential physical questions about this wider class of simulations. Further we have framed the model in a purely statistical mechanics language so that these physical issues, which are related to ones addressed in the wider context of long range interacting systems (as in this book), may be more easily accessible to non-cosmologists.

We thus conclude with a discussion of some specific questions which we can address with this “toy model” and which are of direct relevance to the analogous cosmological simulations. We focus mostly on the issue of discreteness effects, and then briefly discuss the more general, but related, problem of gaining a better theoretical understanding of the non-linear regime of gravitational clustering in these infinite systems.

Discreteness effects and the VP limit

In current cosmological theories of structure formation of the universe the dominant clustering component is purely self-gravitating (“dark”) non-relativistic (“cold”) matter. Theoretically it is described (see e.g. [13]) by a set of Vlasov-Poisson (VP) equations. Although, as mentioned above, there is no rigorous derivation demonstrating the applicability of this limit in this context, it can be understood as an appropriate (mean-field) approximation because there is a separation of length scales, between those characterising the granularity of the *microscopic* physical particles and the *astrophysical* scales

relevant to cosmology. This allows one to view the VP equations as obtained by a coarse-graining over an intermediate scale (see e.g. Ref.[30]).

Solving the VP equations for the 6-dimensional one particle phase space density is, however, not feasible numerically, because of the strong non-linear coupling of scales in the problem. For this reason cosmologists adopt the N -body approach, simulating self-gravitating “macro-particles” in real space. The mass of these unphysical particles are typically 10^{60-80} larger than the supposed mass of the physical particles, and thus the differences between the lattice spacing ℓ and a physical particle separation is at least of order 10^{20} ! The problem of determining the effects of this discretization on the measured clustering is the “discreteness problem”. Until now it is a problem which has been addressed mostly in a very qualitative manner, and there is considerable disagreement in the literature amongst the few authors who have attempted to address it (see Ref. [31] for references). It is now in fact a problem of very practical relevance as extremely precise predictions will be needed in the near future from these simulations in order to compare the current theoretical models with the experimental results produced by several different observational techniques.

Let us just briefly consider the issue within the context of the system we have discussed here. Is the evolution of the system described by the VP equations? While we have shown that the temporal dependence of the asymptotic self-similar behaviour can be derived within this framework, we have seen in the previous section that we can understand aspects of the evolution using a NN approximation, which are clearly incompatible with a VP limit. So what is the VP limit for our system? How do we extrapolate to such a limit?

The class of infinite SL we considered is characterised, as we discussed by just two parameters, which we can choose to be ℓ and δ . Gravity, however, is a scale free and so, dynamically, there is really only one parameter: any two SL with the same δ are in fact equivalent up to a redefinition of the unit of length (which may be chosen equal to ℓ). Thus δ is in fact the single physically relevant parameter. Can we recover a VP approximation for some limiting value of δ ? The answer is clearly in the negative: for any value of δ the NN phase remains³³. It is interesting to note also that as $\delta \rightarrow 0$ the PLT treatment we described becomes valid for an arbitrarily long time. In fact it can be shown[29, 31] that this leads to a divergence of the evolution from the VP limit, given in this case by the linearized fluid evolution. In fact PLT approximates well the fluid evolution on time scales of a few dynamical times, but deviates more and more in time, with an asymptotic divergence³⁴.

To define a VP limit for our infinite system we clearly need a length scale. Indeed as we noted in our discussion of the comparison of the thermodynamic limit and the VP limit above, the VP limit must be one in which the particle density $n_0 \rightarrow \infty$, which corresponds to sending $\ell \rightarrow 0$. We must therefore have another length scale to compare

³³ We have discussed only explicitly the case $\delta < 1$. For any $\delta > 1$ the SL at small scales is identical to a Poisson distribution. In this case the NN phase occurs always at the beginning of the evolution. For more detail, see Refs. [32, 3].

³⁴ We note that this finding makes sense physically as the VP limit is taken at fixed time. We consider here instead the limit in which the particle density is held fixed but the time diverges.

the density with. The scale which can evidently be used is the smoothing length ε , in which case the VP limit is $\ell \rightarrow 0$ at fixed ε (and then, if the limit is indeed defined, $\varepsilon \rightarrow 0$). Studying what we have learnt about the evolution of the SL, we can verify that this limit does indeed lead to the recovery of the fluid limit of the two-phase model we have discussed. Firstly PLT, modified to include the smoothing, can be shown[29, 31] to converge, if $\ell \ll \varepsilon$, to the fluid limit. Further the NN dominated phase we have described disappears also once we reach the regime $\ell \ll \varepsilon$, as the force due to nearby particles is now no longer felt.

The conclusion of this analysis is that to attain strictly the VP limit of such simulations we should work in the parameter range $\ell \ll \varepsilon$. This is *not* the current practice in cosmological simulations(see e.g. Ref. [33]), but does correspond to the conclusions of one of the few controlled numerical studies of the issue in the literature[34]. While the results of (most) simulations, which use $\ell \ll \varepsilon$, are not necessarily grossly wrong, we conclude that numerical extrapolations to the regime $\ell > \varepsilon$ will be necessary to determine how precise they are.

Non-linear structures and quasi-stationary states

We conclude with a brief discussion of the more general question of the nature of non-linear clustering in these systems. In recent years, in parallel with the description of the results of simulations in terms of the two-point correlation properties, cosmologists have developed a different phenomenological description of the results of simulations, using so-called “halo models” (see e.g. [26]). These models following naturally from the “hierarchical” picture of clustering, in which it is initial small overdensities at a given scale which evolve linearly until they collapse, essentially then behaving like independent finite sub-systems. The non-linear structure in simulations can be well described as a collection of such “halos”, which are roughly spherical and approximately virialised structures, with properties very similar to those of structures formed after the “violent relaxation” of a finite gravitating system in an initially unstable configuration. These “halos” are believed to be understood again within the framework of VP equations, and are thus closely analogous to the “quasi-stationary states” (QSS) which have recently been much studied in certain statistical mechanical toy models (see contribution of S. Ruffo in this volume). While they have been characterised in great detail in huge numerical studies, and in particular certain apparently “universal” properties identified (see e.g. Refs. [35, 36, 37, 38]) their physics is not at all understood. In this respect it is interesting to note that a recent detailed study[39] concludes that the framework of Lynden-Bell, exploited recently in the study of such QSS in simple systems, is apparently not useful in predicting their properties.

In the SL system it appears visually, and would be expected from the qualitative features we have seen, that such a halo model description of the clustering could be given. It would be interesting to analyse in this context the nature of these halos, and also their relation to virialized structures formed in finite open systems (e.g. in spherical cold collapse). Further it would be perhaps useful to explore in particular lower dimensional (i.e. two or one) dimensional models which resemble more closely the infinite system we

have discussed in order to simplify and possibly make closer contact with the insights gained, notably about QSS, in the studies of simple statistical mechanical models.

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